Abstract—In this letter, we show how to compute optimal constellations which maximize the capacity of M-PAM input AWGN (MIAWGN) channels with non-equiprobable symbols. We find for binary signaling that the optimal constellations, which minimize the bit error rate, will also maximize the capacity. The rate capacity of MIAWGN channels with non-equiprobable symbols and optimal constellations approaches to $H(X)$ when $E_b/N_0$ becomes large, where $E_b/N_0$ is bit energy-to-noise density ratio and $H(X)$ is the entropy of the source symbols. At the low rate regions (i.e., $R < H(X)/2$) the capacity loss due to non-equiprobable symbols is not significant if the constellations are optimized for the symbol probabilities. However, if we select wrong constellations, the capacity loss could be significant for $M = 2$ and 4. For binary signaling with symbol probabilities of 0.3 and 0.7, 0.72 dB increase of $E_b/N_0$ is required if we use optimal constellations for equiprobable symbols (i.e., $\{x_i = \pm 1\}$). When $M$ becomes large, the shaping effect helps rate maximization and the constellation mismatch has little impact on capacity.

I. INTRODUCTION

Optimal $M$-ary digital communication systems with equiprobable symbols have been well analyzed and used widely in practice, including wireless communications. Good summaries of these works can be found in many books, e.g. [1] and [2]. Recently, these optimal settings have been extended in various directions to accommodate practical needs. Source coding (for example, CELP encoder and decoder for speech signals in [3]) has been commonly applied so that at the output of the source coder the symbols approach equiprobable. However, the assumption of equal symbol probabilities at the output of the source coder is not always satisfied. In many practical applications, due to the sub-optimality of the compression scheme, the bit stream often exhibits a certain amount of redundancy [3], [4]. With these redundancies, the symbol probabilities are not equal [5]. Recently, several papers [6] - [10] have a novel and direct approach to the original problem of optimizing the system with non-equiprobable symbols. In this letter, we aim to determine the optimal constellations which maximize the capacity for M-PAM input AWGN (MIAWGN) channels with non-equiprobable symbols. In Fig. 1 [14] Forney and Ungerboeck showed the capacity of AWGN channels with various M-PAM constellations for equiprobable symbols. In We can view constellations with small $M$ as a special case of constellations with large $M$ and non-equiprobable symbols (for example, 2-PAM as 4-PAM with symbol probabilities of $[0, 0.5, 0.5, 0]$). Here, we consider more general cases.

II. FORMULATION

Let $\{x_i\}$ for $i = 1, \ldots, M$ denote the constellation points for information source $\{I_i\} = 1, 2, \ldots, M$ with probabilities $p_i = Pr(I_i) = Pr(x_i)$, respectively, where $Pr(.)$ denotes the probability. The $i$-th input signal is passed through a AWGN channel and the output is $y = x_i + n$, where $n$ is the noise component with Gaussian distribution of zero mean and variance $\sigma^2 = N_0/2$. Clearly, letting $p(.)$ denote the probability density function, we have $p(y|x_i) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(y-x_i)^2}{2\sigma^2})$, $p(y) = \sum_{i=1}^{M} p(y|x_i) p_i$, and $p(y, x_i) = p(y|x_i) p_i$. According to Shannon channel coding theory, the capacity of this channel is

$$C = \max_{\{p_i, x_i\}} I(X; Y)$$

where the mutual information between $X$ and $Y$ is $I(X; Y) = H(Y) - H(Y|X)$, $H(Y) = \int_Y -p(y) \log_2 p(y)dy$ and $H(Y|X) = \int_Y \sum_{i=1}^M p(y|x_i) \log_2 p(y|x_i)dy$.

If we are free to select $\{x_i\}$ and $\{p_i\}$, the ultimate capacity can be reached when $\{x_i\}$ takes Gaussian distribution and the capacity in bits per dimension is $C = \frac{1}{2} \log_2(1 + 2Rb/N_0)$ [14] where $R$ is the code rate in bits per dimension and $E_b/N_0 = (\frac{R}{2\sigma^2})$, where $E_s = E(x^2) = \sum_{i=1}^M p_i x_i^2$. From $R < C = \frac{1}{2} \log_2(1 + 2Rb/N_0)$, we obtain

$$\frac{E_b}{N_0} > \frac{2^{2R} - 1}{2R} = \frac{E_b}{N_{0, AWGN}}$$

For the rate half case, i.e., $R = C_{AWGN} = 1/2$, we have $\frac{E_b}{N_{0, AWGN}} = 1$ (i.e., 0 dB), where $E_s = 1$ and $\sigma^2 = 1$.

If we fix the constellations as $\{x_i\} = \pm 1, \pm 3, \ldots, \pm M$, the optimal value is reached when $p_i = 1/M$ (i.e., equiprobable symbols) and the capacity is

$$\frac{E_b}{N_0} > \left( \frac{E_s}{C_{MIAWGN}N_0} \right) = \frac{E_b}{N_{0, MIAWGN}}$$

where

$$C_{MIAWGN} = I(X; Y)|_{\{x_i\} = \pm 1, \pm 3, \ldots, \pm M, p=1/M}$$

When $\sigma = 0.97869$, we have $R = C_{BIAWGN} = 0.5$ and $\frac{E_b}{N_{0, BIAWGN}} = 0.187$ dB.

For general cases with constraint on constellation, methods given in [11], [12], and [13] can quickly find the optimal values

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of \( \{p_i\} \) and the capacity. For very large \( M \)-ary QAM constellation, the shaping gain is limited to 1.53 dB [14], which can be achieved using trellis shaping for conventional TCM [15], iterative decoding for parity concatenated TCM systems [16], and signal shaping for bit-interleaved coded modulation [17]. However, all these works are focused on equiprobable source symbols. The trellis shaping is done by expanding constellation first, then assigning equiprobable constellation points to achieve the gain. If we have imperfect source coding that produces a fixed, but non-equal \( \{p_i\} \), we have just to know how to compute optimal constellations which minimize symbol error probabilities for several types of simple receivers [7]-[10]. Before we extend these constellations for systems with error control coding, we need to understand their impacts on channel capacity. How do we optimize constellation \( \{x_i\} \) to maximize the capacity with non-equiprobable symbols? Can we achieve trellis shaping without constellation expansion for these cases? We have not found any prior work that answers these questions. In this letter, let us start to answer this question by focusing on optimal constellation design for \( M \)-PSK input AWGN channels.

III. OPTIMIZATION

Letting \( z = y/\sqrt{N_0} \) and \( g_i = x_i/\sqrt{N_0} \), we can simplify (1) as

\[
I(X;Y) = \sum_{i=1}^{M} \int_{-\infty}^{\infty} \frac{p_i}{\sqrt{\pi}} e^{-z-g_i^2} e^{-(z-g_i)^2} dz
\]

In order to optimize \( I(X;Y) \) for given \( p_i \) and \( \sum_{i=1}^{M} p_i g_i^2 = \frac{E_b}{N_0} \), we define the objective function

\[
\Omega = I(X;Y) + \lambda \left( \sum_{i=1}^{M} p_i g_i^2 - \frac{E_b}{N_0} \right)
\]

where \( \lambda \) is a Lagrange multiplier. We find the optimal \( \{g_i\} \) and \( \lambda \) by taking the derivatives, \( d\Omega/dg_i = 0 \) and \( d\Omega/d\lambda = 0 \) for \( i = 1, 2, ..., M \), where

\[
\frac{d I(X;Y)}{d g_i} = \sum_{i \neq k}^{M} \int_{-\infty}^{\infty} 2p_ip_k \left[ \frac{(g_k - g_i - z)\exp(-z^2 + (g_k - g_i)(2z - g_k + g_i))}{\sqrt{\pi}\ln 2 \sum_{j=1}^{M} p_j \exp((g_j - g_i)(2z - g_j + g_i))} + \frac{(g_k - g_i + z)\exp(-z^2 + (g_k - g_i)(2z - g_k + g_i))}{\sqrt{\pi}\ln 2 \sum_{j=1}^{M} p_j \exp((g_j - g_i)(2z - g_j + g_k))} \right] dz \]

(7)

Now let us study several propositions.

**Proposition 1:** For \( M = 2 \), optimal constellation satisfies \( p_1g_1 + p_2g_2 = 0 \).

**Proof:** Taking \( d\Omega/dg_i = 0 \) for \( i = 1, 2 \), we have

\[
\frac{d I(X;Y)}{d g_i} + 2\lambda p_i = 0
\]

(8)

We only need to show \( \frac{d I(X;Y)}{d g_1} = \frac{d I(X;Y)}{d g_2} \) to complete the proof.

Rewriting (7), we obtain

\[
\frac{d I(X;Y)}{d g_1} = \int_{-\infty}^{\infty} \frac{p_1p_2\exp(-z+g_1-g_2)^2}{\sqrt{\pi}\ln 2} \left( (p_1 + p_2\exp((g_1-g_2)(2z-g_1+g_2)) \right. dx
\]

\[
- \int_{-\infty}^{\infty} \frac{2p_1p_2\exp(-z+g_1-g_2)^2}{\sqrt{\pi}\ln 2} \left( p_2 + p_1\exp((g_2-g_1)(2z-g_1+g_2)) \right) dx
\]

\[
= - \frac{d I(X;Y)}{d g_2}
\]

This proposition shows that the constellation minimizing bit error rates for the coherent matched filter receiver (see [7] and [9]) maximizes also the capacity. Using this result, we can compute the optimal values of \( g_1o = \sqrt{\frac{E_b}{p_1^2} \frac{p_2}{N_0}} \), \( g_2o = -\sqrt{\frac{E_b}{p_2^2} \frac{p_1}{N_0}} \), and \( G_o = g_1o - g_2o = \sqrt{\frac{E_b}{N_0} \left( \frac{p_1}{p_2} + \frac{p_2}{p_1} \right)} \).

**Proposition 2:** \( I(X;Y) \leq H(X) \), where \( H(X) = \sum_{i=1}^{M} -p_i \log_2(p_i) \). Furthermore, when \( E_b/N_0 \to \infty \), the capacity \( I(X;Y) \) converges to \( H(X) \) regardless signal constellations.

**Proof:** Putting \( \{g_i\} \) (with distinct \( M - J \) values) in order from least to greatest, we have a new sequence of a total of \( J \leq M \) elements, \( \{g'_1, g'_2, ..., g'_J\} \), where \( g'_i < g'_j \) if \( i < j \). Ordering \( \{p_i\} \) accordingly and combining probabilities together for those points with same \( x_i \), i.e., if the \( k \)-th distinct value is \( g_i \) and \( g_i = g_j \), then \( p'_k = p_i + p_j \). It is easy to show that \( H(X) = H(X') \), \( H(X;Y) = H(X';Y) \), where

\[
I(X';Y) = \sum_{i=1}^{J} \int_{-\infty}^{\infty} -p'_i \exp(-z^2/\sqrt{\pi}) \log_2 \left( p'_i + \sum_{j=1}^{M} p_j \exp((g'_j - g'_i)(2z - g'_j + g'_i)) \right) dz
\]

\[
\leq \sum_{i=1}^{J} \int_{-\infty}^{\infty} -p'_i \exp(-z^2/\sqrt{\pi}) \log_2(p'_i) dz = H(X')
\]

(10)

\[
\lim_{\frac{E_b}{N_0} \to \infty} I(X';Y) = \lim_{\frac{g'_i}{g'_j} \to \infty} I(X';Y) = H(X')
\]

(11)

The second proposition shows that \( M \)-ary constellation may reduce to a small \( J \)-ary constellation without losing capacity and at high \( E_b/N_0 \) regions the capacity will converge to \( H(X) \) regardless of selecting of constellations. But, in practice we are more interested in low \( E_b/N_0 \) regions where the capacities of MIAWGN channels are fairly close to the ultimate AWGN channel capacity.

Finally, using Matlab to solve equations, we can compute capacities with optimal constellation and mismatched constellation using

\[
C = I(X;Y)|_{\{g'_i\}, \{p_i\}}
\]

(12)
and calculate the minimum required $E_b/N_0$ for a given rate.

IV. NUMERICAL RESULTS

Firstly, we consider $M = 2$ and $\{p_1 = 0.3, p_2 = 0.7\}$. In Fig. 1, we plot the rate as a function of $E_b/N_0$ for five cases: (a) AWGN channel, i.e., $2^{2R} - 1$, (b) BIAWGN channel with equiprobable symbols and optimal constellations, (c) BIAWGN channel with equiprobable symbols and constellations $\{x_i\} = \pm 1$, (d) BIAWGN channel with non-equiprobable (p=0.3) symbols and optimal constellation, i.e., $p_1 g_1 = -p_2 g_2$, (e) BIAWGN channel with non-equiprobable symbols (p=0.3) and constellation $\{x_i\} = \pm 1$. As showed in Proposition 2, at high $E_b/N_0$, the rate of cases (b) and (c) converges to 1, the rates of cases (d) and (e) converge to $H(p)$. At low $E_b/N_0$, the rate with optimal constellation (case(d) converges to that of case (b), but the rate of mismatched constellation is significant. In Fig. 2, we plot the rate as a function of $E_b/N_0 = E_s/(RN_0)$ for these five cases. The loss of $E_b/N_0$ due to mismatched at the low rate region is about 0.72 dB.

In Fig. 3, we repeat the computations for $M = 4$ and $\{p_i\} = [0.09, 0.21, 0.49, 0.21]$. The results show the rates for case (d) of non-equiprobable symbols are higher than those for cases (b) and (c) due to the shaping effect (i.e., assigning high probable symbols to constellation points of lower energy). The loss in $E_b/N_0$ due to mismatched constellation becomes smaller, 0.51 dB. The normalized optimal $\{g_i\}$ values (i.e., $E\{g_i^2\} = 1$) are [-2.021, -0.742, 0.017, 1.569], [-1.951, -0.879, 0.079, 1.531], and [-2.094, -0.949, 0.203, 1.373] for $E_b/N_0 = -0.922, 0.138, 11.98 dB$, respectively. Clearly, when $E_b/N_0$ becomes large, it approaches the equal space constellation. We have shown that similar behavior for optimal constellations minimizing the bit error rate in [9]. But, constellations at lower $E_b/N_0$ regions are far more important in practice, particularly with error control coding.

In Fig. 4, we repeat the computations for $M = 8$ and $\{p_i\} = [0.027, 0.063, 0.147, 0.343, 0.147, 0.063, 0.063]$. The results show that the shaping effect makes non-equiprobable cases slight more attractive than equiprobable cases. The loss in $E_b/N_0$ due to constellation mismatch becomes very small.

If we take a look at Fig. 1 in [14], then our results make sense. We select different constellations for conveniences in practice, but we have never allowed to use 2-PAM to receive 4-PAM signals. So, when imperfect source coding generates non-equiprobable symbols, we cannot ignore and treat them as equiprobable cases for MPSK signaling with small values of $M$.

V. CONCLUSIONS AND DISCUSSIONS

In this letter, we have shown how to compute optimal constellations which maximize the capacity of MPSK input AWGN (MIAWGN) channels with non-equiprobable symbols. We found for BPSK signaling, the constellation that minimizes the bit error rate also maximizes the capacity. The rate capacity of MIAWGN channels with non-equiprobable symbols and optimal constellations approaches to $H(X)$ when $E_b/N_0$ or $E_s/N_0$ becomes large. At the low rate regions, (i.e., $R < H(X)/2$), the capacity loss due to non-equiprobable symbols is not significant, as long as the constellations are optimized for the symbol probabilities. However, if we select wrong constellations, the capacity loss could be significant for small $M$. As an example, when $p = 0.3$, it would cost 0.72 dB reduction in $E_b/N_0$ by choosing constellation optimized for equiprobable symbols (i.e., $\{x_i = \pm 1\}$) instead of optimal constellation for non-equiprobable symbols. When $M$ becomes large, the shaping effect helps rate maximization and the constellation mismatch has much milder impact on capacity.

There are still many unanswered questions. For large $M$, trellis shaping [15] can effectively map source symbols into constellation points approaching to Gaussian distribution. But, we do not know yet how to design capacity approaching codes for non-equiprobable symbol cases for $M = 2$ and 4. Furthermore, we need to work out how to use optimal constellations in practical system design. With exponentially increasing in the cost of spectrum, we may need to further reduce imperfectness. Our efforts could lead to discover ways to achieve a better complexity trade-off between source and channel coding.

REFERENCES


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Fig. 1. $R$ of systems with $M=2$, equal and non-equiprobable ($\{p_i\} = [0.3, 0.7]$) symbols as a function of $E_s/N_0$.

Fig. 2. $R$ of systems with $M=2$, equal and non-equiprobable ($\{p_i\} = [0.3, 0.7]$) symbols as a function of $E_b/N_0$.

Fig. 3. $R$ of with $M=4$, equal and non-equiprobable ($\{p_i\} = [0.09, 0.21, 0.49, 0.21]$) symbols as a function of $E_b/N_0$.

Fig. 4. $R$ of with $M=8$, equal and non-equiprobable ($\{p_i\} = [0.027, 0.063, 0.147, 0.343, 0.147, 0.147, 0.063, 0.063]$) symbols as a function of $E_b/N_0$. 